Distinguished models of intermediate Jacobians

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The quest for the phantom

Mazur's Question

$X/\mathbb{Q}$ a smooth projective threefold, $h^{3,0} = h^{0,3} = 0$.
Is there an abelian variety $A/\mathbb{Q}$:

$$H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1)) \cong H^1(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)?$$

Such an A is called a phantom.

Joint work with Sebastian Casalaina-Martin (Boulder) and Charles Vial (Bielefeld).

Weights

$Y/\mathbb{Q}$ smooth, projective.

- $H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ pure of weight r:

$$\left|\mathrm{Fr}_p \,|H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\right| = \sqrt{p^r}.$$

- $H^r(Y_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(j))$ is pure of weight $r - 2j$.

Hodge numbers

$Y/\mathbb{C}$ smooth, projective.

- $H^r(Y(\mathbb{C}),\mathbb{Q})$ has Hodge structure of weight r:

$$H^r(Y(\mathbb{C}),\mathbb{Q})\otimes\mathbb{C} = \oplus_{p+q=r} H^{p,q}(Y)$$
$$H^{p,q}(Y) = H^q(Y(\mathbb{C}),\Omega^p_Y)$$
$$h^{p,q}(Y) = \dim H^{p,q}(Y)$$

- Ex: $\dim Y = 3$

$$\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{31} & & h^{22} & & h^{13} & \\
 & & h^{32} & & h^{23} & & \\
 & & & h^{33} & & &
\end{array}$$

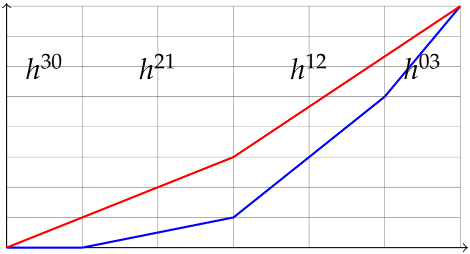
Newton over Hodge

$X/\mathbb{Z}_p$ smooth, projective, good reduction.

- $\mathrm{NP}(X,r)$ Newton polygon of Fr on $H^r_{\mathrm{dR}}(X_{\mathbb{Q}_p}) \cong H^r_{\mathrm{cris}}(X_p)$.
- $\mathrm{HP}(X,r)$ r^{th} Hodge polygon, vertices $(\sum_{0\le j\le k} h^{r-j,j}, \sum_{0\le j\le k} jh^{r-j})$.

Theorem (Mazur)

$\mathrm{NP}(X,r)$ *lies on or above* $\mathrm{HP}(X,r)$.



Divisibility

Corollary

If $h^{30}(X) = 0$*, then each eigenvalue of Frobenius on* $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ *is an algebraic integer of size* $\sqrt{p}$*.*

Proof.

- $\mathrm{NP}(X,3)$ over $\mathrm{HP}(X,3)$ implies all slopes of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ are ≥ 1.
- $\implies$ each eigenvalue α of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ divisible by p
- $\implies$ each eigenvalue α/p of Fr_p on $H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ is algebraic integer of size $\sqrt{p}$.

$\square$

$H^3(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$ could come from an abelian variety

Jacobians

Jacobians as Phantoms

If X/K smooth projective, then Pic^0_X is a phantom in degree 1.

From Kummer sequence

$$1 \longrightarrow \boldsymbol{\mu}_N \longrightarrow \mathcal{O}_X^\times \xrightarrow{[N]} \mathcal{O}_X^\times \longrightarrow 1$$

get

$$0 \longrightarrow H^1(X_{\overline{K}}, \boldsymbol{\mu}_N) \longrightarrow H^1(X_{\overline{K}}, \mathcal{O}_X^\times) \longrightarrow H^1(X_{\overline{K}}, \mathcal{O}_X^\times)$$

so

$$H^1(X_{\overline{K}}, \mathbb{Z}/N(1)) \cong \ker\left(H^1(X_{\overline{K}}, \mathcal{O}_X^\times) \to H^1(X_{\overline{K}}, \mathcal{O}_X^\times)\right) \cong \mathrm{Pic}^0_X[N](\overline{K}).$$

Complex Jacobians

$X/\mathbb{C}$ smooth projective

Exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

gives

$$H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^\times) \longrightarrow H^2(X,\mathbb{Z})$$

$$\cong \operatorname{Pic}_X(\mathbb{C})$$

$$\subseteq \operatorname{Pic}^0_X(\mathbb{C})$$

and so

$$\operatorname{Pic}^0_X(\mathbb{C}) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}.$$

Intermediate Jacobians

$$\begin{aligned}\mathrm{Pic}^0_X(\mathbb{C}) &\cong \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}\\ &\cong \mathrm{Fil}^1 H^1(X,\mathbb{C})\backslash H^1(X,\mathbb{C})/H^1(X,\mathbb{Z}).\end{aligned}$$

More generally, intermediate Jacobians are

$$J^{2n+1}(X) = \mathrm{Fil}^{n+1}\backslash H^{2n+1}(X,\mathbb{C})/H^{2n+1}(X,\mathbb{Z}).$$

If $H^{2n+1}(X,\mathbb{C})$ has Hodge level one, then

- $H^{2n+1} = H^{n+1,n} \oplus H^{n,n+1}$;
- Complex torus $J^{2n+1}(X)$ is actually an abelian variety.

Complete intersections: Deligne

Theorem (Deligne)

Suppose $X/\mathbb{Q}$ a complete intersection of dimension $2n+1$, and $H^{2n+1}(X, \mathbb{C})$ has Hodge level one. Then $J^{2n+1}(X_{\mathbb{C}})$ descends to an abelian variety $J/\mathbb{Q}$, and J is a phantom for X.

Idea

- Monodromy action on universal $\mathcal{J}^{2n+1}(\mathcal{X})$ over Hilbert scheme is irreducible.
- Descent.

Coniveau

X/K smooth projective.

$$N^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \subseteq \widetilde{N}^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \subseteq H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$$

- $N^r H^i$ from $Y \hookrightarrow X$ of codim r.
- $\widetilde{N}^r H^i$ is maximal $M \subset H^i$; $M(r)$ effective.

Generalized Tate Conjecture

$$N^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell) = \widetilde{N}^r H^i(X_{\overline{K}}, \mathbb{Q}_\ell).$$

Abel–Jacobi

X/K smooth projective.

- $\mathrm{CH}^r(X) = \{\text{codim } r \text{ cycles}\} \,/\, \{\text{rat equiv}\}$ Chow group.
- $\mathrm{A}^r(X) \subset \mathrm{CH}^r(X)$ algebraically trivial cycles.

If $X/\mathbb{C}$, have Abel–Jacobi map

$$\mathrm{A}^{n+1}(X) \xrightarrow{\mathrm{AJ}} J^{2n+1}(X)$$

- $J_a^{2n+1}(X) := \mathrm{im}(\mathrm{AJ})$ is an abelian variety.
- $H^1(J_a^{2n+1}) = \mathrm{N}^n\, H^{2n+1}(X)(n)$.

Main result

Theorem (A.–C.-M.–V.)

X/K a smooth projective variety over a subfield of $\mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Then there exist an abelian variety J/K and cycle $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times X)$ such that:

$$J_{\mathbb{C}} = J_a^{2n+1}(X_{\mathbb{C}});$$

the Abel–Jacobi map

$$\mathrm{A}^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J(\mathbb{C})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant; and

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \xhookrightarrow{\Gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

is a split inclusion with image $\mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Lemma

There exist:

- C/K *a smooth projective geometrically irreducible curve;*
- $\gamma \in \mathrm{CH}^{n+1}(C \times X)$ *a correspondence on* $C \times X$*;*

such that the induced map is surjective:

$$H^1(C_{\overline{K}}, \mathbb{Q}_\ell) \overset{\gamma_*}{\twoheadrightarrow} \mathrm{N}^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)).$$

Strategy

- $\exists f : Y \hookrightarrow X/K$, codim n,

$$f_* H^1(Y_{\overline{K}}, \mathbb{Q}_\ell) = \mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell)(n).$$

- Bertini: $C \hookrightarrow Y$ a curve, $H^1(Y) \hookrightarrow H^1(C)$.
- γ Construct a correspondence via

$$H^1(C) \hookrightarrow H^{2d_Y-1}(Y) \xrightarrow[(L^{d_Y})^{-1}]{\sim} H^1(Y) \longrightarrow H^{2n+1}(X)$$

(Only middle arrow difficult; Lefschetz standard conjecture in degree one.)

Can take C geometrically irreducible using:

- $\beta : C \to \mathrm{Pic}^0_C$ inducing isomorphism on $H^1(\cdot, \mathbb{Q}_\ell)$;
- Bertini for geometrically irreducible variety Pic^0_C.

We have

$$J^1(C_{\mathbb{C}}) \xrightarrow{\gamma_*} J_a^{2n+1}(X_{\mathbb{C}}).$$

- $J^1(C_{\mathbb{C}}) = (\mathrm{Pic}^0_C)_{\mathbb{C}}$ has a distinguished model over K.
- Use this and γ_* to obtain model for $J_a^{2n+1}(X_{\mathbb{C}})$.

$\mathbb{C}/\overline{K}$

- $\mathbb{C}/\overline{K}$ is a regular extension of fields.
- $\underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}}) := \operatorname{tr}_{\mathbb{C}/\overline{K}}(J_a^{2n+1}(X_{\mathbb{C}}))$ is "largest" sub-abelian variety defined over $\overline{K}$.

Rigidity:

$$\operatorname{Hom}_{\overline{K}}(J(C)_{\overline{K}}, \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})) = \operatorname{Hom}_{\mathbb{C}}(J(C_{\overline{K}})_{\mathbb{C}}, J_a^{2n+1}(X_{\mathbb{C}})).$$

Get surjection

$$J(C_{\overline{K}}) \twoheadrightarrow \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})$$

of abelian varieties over $\overline{K}$.

$\overline{K}/K$

- Need to show

$$J(C_{\overline{K}}) \xrightarrow{\gamma_*} \underset{=a}{J}^{2n+1}(X_{\mathbb{C}})$$

  descends to K.
- Suffices to show all

$$(\ker \gamma_*)[N](\overline{K})$$

  stable under $\mathrm{Gal}(K)$.

Strategy suggested to us by Gabber.

Follow the arrows

$$J(C_{\overline{K}})[N] \longrightarrow \underline{J}_a^{2n+1}(X_{\mathbb{C}})[N]$$

Follow the arrows

$$\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{J}^{2n+1}_{a}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] & J^{2n+1}_{a}(X_{\mathbb{C}})[N]
\end{tikzcd}$$

Follow the arrows

$$
\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, hook] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1))
\end{tikzcd}
$$

Follow the arrows

$$\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{\underline{J}}_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, hook] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d, "\sim"'] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1_{\text{ét}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] & H^{2n+1}_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1))
\end{tikzcd}$$

Follow the arrows

$$
\begin{tikzcd}
J(C_{\overline{K}})[N] \arrow[r] \arrow[d, "\sim"'] & \underline{\underline{J}}^{2n+1}_{a}(X_{\mathbb{C}})[N] \arrow[d, "\sim"] \\
J(C_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\sim"'] & J^{2n+1}_{a}(X_{\mathbb{C}})[N] \arrow[d, hook] \\
H^1_{\mathrm{an}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d, "\sim"'] & H^{2n+1}_{\mathrm{an}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1_{\text{ét}}(C_{\mathbb{C}}, \mathbb{Z}/N(1)) \arrow[r] \arrow[d] & H^{2n+1}_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/N(n+1)) \arrow[d, "\sim"] \\
H^1(C_{\overline{K}}, \mathbb{Z}/N(1)) \arrow[r, "\gamma_*"'] & H^{2n+1}(X_{\overline{K}}, \mathbb{Z}/N(n+1))
\end{tikzcd}
$$

Models

- Since $\ker(J^1(C)_{\overline{K}} \to \underline{J}_a^{2n+1}(X_{\mathbb{C}}))$ stable under $\mathrm{Gal}(K)$, we have a model J/K for $J_a^{2n+1}(X_{\mathbb{C}})$.
- How do we know this is the right model?

Recall the Abel–Jacobi map

$$A^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J_a^{2n+1}(X_{\mathbb{C}}).$$

Lemma

The model J/K of $J_a^{2n+1}(X_{\mathbb{C}})$ makes AJ $\mathrm{Gal}(K)$*-equivariant on torsion.*

Recall the Abel–Jacobi map

$$A^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J_a^{2n+1}(X_{\mathbb{C}}).$$

Lemma

The model J/K of $J_a^{2n+1}(X_{\mathbb{C}})$ makes AJ $\mathrm{Gal}(K)$*-equivariant on torsion.*

$$\begin{tikzcd}
A^{n+1}(X_{\mathbb{C}})[N] \arrow[r] \arrow[d, "\mathrm{Lecomte}"', "\sim"] & J_a^{2n+1}(X_{\mathbb{C}})[N] \arrow[d, "="] \\
A^{n+1}(X_{\overline{K}})[N] \arrow[r] \arrow[d, "\mathrm{Bloch}"', "\lambda^{n+1}"] & J_{\overline{K}}[N] \arrow[dl, hook] \\
H^{2n+1}(X_{\overline{K}}, \mathbb{Z}/N(n+1)) &
\end{tikzcd}$$

Corollary

J is a phantom for X in degree $2n+1$.

- Still want to show

$$\mathrm{A}^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J(\mathbb{C})$$

  is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant.
- Rigidity fails for non-torsion points (on abelian varieties) and cycles (on arbitrary varieties).

Key Tool

$\mathrm{AJ} : A^{n+1}(X_{\mathbb{C}}) \to J_a^{2n+1}(X)(\mathbb{C})$ is *regular* (in the sense of Samuel).

Regular maps

- $X/k = \bar{k}$, A/k an abelian variety.
- An abstract group homomorphism

$$\mathrm{A}^i(X) \xrightarrow{\phi} A(k)$$

is regular if for every pointed variety (T, t_0), and every family of cycles $Z \in \mathrm{CH}^i(T \times X)$, the map of sets

$$\begin{array}{ccccc} T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\ t & \longmapsto & [Z_t] - [Z_{t_0}] & & \end{array}$$

Regular maps

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is regular if for every pointed variety (T, t_0), and every family of cycles $Z \in \mathrm{CH}^i(T \times X)$, the map of sets is induced by a morphism

$$\begin{array}{ccccc} T(k) & \xrightarrow{w_Z} & \mathrm{A}^i(X) & \xrightarrow{\phi} & A(k) \\ t & \longmapsto & [Z_t] - [Z_{t_0}] & & \\ T & & \overset{\psi_Z}{\dashrightarrow} & & A \end{array}$$

Ω/k

Lemma

Ω/k an extension of algebraically closed fields of characteristic zero, X/k smooth projective, A/Ω an abelian variety,

$$\mathrm{A}^i(X_\Omega) \xrightarrow{\phi} A(\Omega)$$

regular and surjective. Then $A = (\underline{\underline{A}})_\Omega$; $\phi = (\underline{\underline{\phi}})_\Omega$; and

$$\mathrm{A}^i(X) \xrightarrow{\underline{\underline{\phi}}} \underline{\underline{A}}(k)$$

is regular and surjective.

Key Idea

Use rigidity; $\mathrm{A}^i(X_\Omega)[N] \cong \mathrm{A}^i(X_{\overline{K}})[N]$.

$\overline{K}/K$

Proposition

K perfect, X/K smooth and projective, A/K an abelian variety. Suppose

$$\mathrm{A}^i(X_{\overline{K}}) \xrightarrow{\phi} A(\overline{K})$$

is regular and surjective.
If $\phi[\ell^n]$ is $\mathrm{Gal}(K)$-equivariant for all n, then ϕ is $\mathrm{Gal}(K)$-equivariant.

Key Idea

For test varieties (T, t_0), abelian varieties are enough.

Weil's lemma

Algebraically trivial cycles are witnessed by abelian varieties:

Lemma

Let X/K be a scheme of finite type over a field, and let $\alpha \in \mathrm{A}^i(X_{\overline{K}})$ be an algebraically trivial cycle class.
Then there exist an abelian variety B/K, a cycle class $Z \in \mathrm{CH}^i(B \times X)$, and a $t \in Z(\overline{K})$ such that

$$\alpha = [Z_t] - [Z_0].$$

- Weil (and Lang) prove this for $K = \overline{K}$.
- Their proof breaks down over arbitrary K; may not be enough Brill-Noether generic K-rational points.

For regular maps, $\mathrm{Gal}(K)$-equivariance on torsion implies equivariance:

- Weil's lemma: Find B/K abelian variety, $Z \in \mathrm{CH}^i(B \times X)$,

$$B(\overline{K}) \xrightarrow{w_Z} \mathrm{A}^i(X_{\overline{K}}) \longrightarrow A(\overline{K})$$

  surjective.
- On torsion, have

$$B(\overline{K})[\ell^\infty] \xrightarrow{w_Z[\ell^\infty]} \mathrm{A}^i(X_{\overline{K}})[\ell^\infty] \xrightarrow{\phi[\ell^\infty]} A(\overline{K})[\ell^\infty]$$

- $\phi[\ell^\infty]$ $\mathrm{Gal}(K)$-equivariant by hypothesis.
- $w_Z[\ell^\infty]$ is $\mathrm{Gal}(K)$-equivariant since $Z/K, 0 \in B(K)$.

So $\psi : B_{\overline{K}} \to A_{\overline{K}}$ descends to K.

Consequence

Corollary

If $K \subset \mathbb{C}$, *then* $\mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J(\mathbb{C})$ *is* $\mathrm{Aut}(\mathbb{C}/K)$*-equivariant.*

Transport de structure

Construction of J is functorial in $K \hookrightarrow \mathbb{C}$:

Lemma

If $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, then

$$J_a^{2n+1}((X_{\mathbb{C}})^\sigma) \cong J_a^{2n+1}(X_{\mathbb{C}})^\sigma.$$

Idea

If $\Gamma \in \mathrm{CH}(J \times X)$ witnesses J as the algebraic intermediate Jacobian of X, then Γ^σ does the same for J^σ and X^σ.

Classification

$X_n(a_1, \cdots, a_d) \subset \mathbb{P}^{n+d}$ a smooth complete intersection of dimension n, multidegree $\underline{a}$.

Rapoport's Classification

A smooth complete intersection has Hodge level one if and only if it belongs to the following list:

$X_n(2,2)$	intersection of two quadrics in $\mathbb{P}^{n+2}$;
$X_n(2,2,2)$	intersection of three quadrics;
$X_3(3)$	cubic threefold;
$X_3(2,3)$	a threefold, realized as the intersection of a quadric and cubic;
$X_5(3)$	cubic fivefold;
$X_3(4)$	quartic threefold.

Period maps for Hodge level one

Distinguished models give new proof of:

Theorem (Deligne)

Let $\mathcal{V}$ be a moduli space of complete intersection varieties of Hodge level one. The period map

$$\mathcal{V}(\mathbb{C}) \longrightarrow \mathcal{A}_{g(\mathcal{V})}(\mathbb{C})$$

is induced by a morphism

$$\mathcal{V}_{\mathbb{Q}} \longrightarrow \mathcal{A}_{g(\mathcal{V}),\mathbb{Q}}$$

over $\mathbb{Q}$.

From points to period maps

Proof.

- If $X \in \mathcal{V}(\mathbb{C})$,

$$\mathrm{CH}_0(X)_{\mathbb{Q}}, \cdots, \mathrm{CH}_{n-1}(X)_{\mathbb{Q}}$$

  spanned by linear sections (Otwinoska).
- Decomposition of the diagonal; $\mathrm{A}^n(X) \to J^{2n+1}(X)$ surjective, so $J^{2n+1}(X) = J_a^{2n+1}(X)$ (Bloch-Srinivas).
- Since $J^{2n+1}(X^\sigma) = J^{2n+1}(X)^\sigma$,

$$\left\{(X, J^{2n+1}(X))\right\} \subset (\mathcal{V} \times \mathcal{A}_{g(\mathcal{V})})(\mathbb{C})$$

  is stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, and the period map descends.

$\square$

Specialization

- $R/\mathbb{C}$ a discrete valuation ring
- $X \to S = \operatorname{Spec} R$ a smooth projective scheme.
- $J(X_\eta) = J_a^{2n+1}(X_\eta)$, etc.

Then:

- $J(X_\eta)$ extends to an abelian scheme $\underline{J}(X_\eta)/S$;
- There is a specialization map

$$\underline{J}(X_\eta)_0 \hookrightarrow J(X_0)$$

Jumps in J

In general, $\underline{J}(X_\eta)_0 \to J(X_0)$ is not surjective.

Example

- E a CM field with $[E:\mathbb{Q}] = 6$, $[\widetilde{E} : \widetilde{E^{(+)}}] = 8$.
- $X \to S$ an abelian threefold with:
  - $\mathrm{End}(X_{\overline{\eta}}) \cong \mathbb{Z}$;
  - $\mathrm{End}(X_0) \cong \mathcal{O}_E$.

  Then (Tankeev)
  - $\dim J_a^3(X_{\overline{\eta}}) = \frac{1}{2} \dim \mathrm{N}^1 H^3(X_{\overline{\eta}}, \mathbb{Q}_\ell) = 3$;
  - $\dim J_a^3(X_0) = \frac{1}{2} \dim \mathrm{N}^1 H^3(X_0, \mathbb{Q}_\ell) = 9$.

In general, $s \mapsto \dim J_a^{2n+1}(X_s)$ is upper semicontinuous.

Jumping locus

- $S/\mathbb{Q}$ reduced and irreducible, with generic point η.
- $X \to S$ a smooth projective scheme.

Set-theoretically define the jumping locus

$$S^{\text{jump}} = S^{\text{jump}}(X, n) = \left\{ s \in S(\mathbb{C}) : \dim J_a^{2n+1}(X_s) > \dim J_a^{2n+1}(X_{\eta_{\mathbb{C}}}) \right\}.$$

Modeled on Hodge locus (Cattani-Deligne-Kaplan).

S^{jump} is algebraic

Proposition *(provisional)*

S^{jump} descends to $\mathbb{Q}$ as a countable union of algebraic subvarieties.

S^{jump} is algebraic

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S^{jump} descends to $\mathbb{Q}$ as a countable union of algebraic subvarieties.

Idea

- S^{jump} is (complex-analytically) locally a countable union of closed analytic subsets.
- Since $J(X^\sigma) = J(X)^\sigma$, S^{jump} is stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$.

Derived equivalence

- X/K smooth projective variety over a field.
- $\mathsf{D}(X)$ bounded derived category of coherent sheaves on X.

$\mathsf{D}(X)$ encodes lots of information about X.

Sample Theorem [Orlov]

If X and Y are smooth projective varieties over K with ample (anti-)canonical bundle, and if $\mathsf{D}(X) \cong \mathsf{D}(Y)$, then $X \cong Y$.

Categorical invariants

If $\mathsf{D}(X) \cong \mathsf{D}(Y)$, then, e.g.,

- $\dim X = \dim Y$;
- $\kappa(X) = \kappa(Y)$ (Orlov)
- $$\oplus H^{2i}(X)(i) \cong \oplus H^{2i}(Y)(i)$$
  $$\oplus H^{2i+1}(X)(i) \cong \oplus H^{2i+1}(Y)(i)$$

  where $H^\bullet$ is some Weil cohomology with weights (Mukai).
- $\mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \cong \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$ (Rouquier).

Jacobians

Theorem

Let X and Y be smooth, projective varieties over a field K. If $\mathsf{D}(X) \cong \mathsf{D}(Y)$, then $(\mathrm{Pic}^0_X)_{red}$ and $(\mathrm{Pic}^0_Y)_{red}$ are isogenous over K.

- For $K = \mathbb{C}$, Popa–Schnell.
- For K arbitrary, Honigs–A.–C.-M.–V.

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Corollary (Honigs)

If X and Y are derived equivalent threefolds over $\mathbb{F}_q$, then

$$Z_X(T) = Z_Y(T).$$

Total intermediate Jacobians

Theorem

Let X and Y be smooth projective varieties over a field $K \subset \mathbb{C}$ with $\mathsf{D}(X) \cong \mathsf{D}(Y)$. Then the total algebraic intermediate Jacobians

$$\underline{J}_a(X) = \oplus \underline{J}_a^{2n+1}(X_{\mathbb{C}}) \text{ and } \underline{J}_a(Y) = \oplus \underline{J}_a^{2n+1}(Y_{\mathbb{C}})$$

are isogenous over K.

Threefolds

Corollary

If $\dim X = \dim Y = 3$ *and* $\mathsf{D}(X) \cong \mathsf{D}(Y)$, *then*

$$\underline{J}^3_a(X_{\mathbb{C}}) \sim \underline{J}^3_a(Y_{\mathbb{C}}).$$

Idea

Use:

- Popa–Schnell: $J^1(X) \sim J^1(Y)$;
- Auto-duality: $J^5(X) \sim J^5(Y)$;
- Poincaré reducibility: "cancel" from $J_a(X)$ and $J_a(Y)$.

Threefolds

Corollary

If $\dim X = \dim Y = 3$ *and* $\mathsf{D}(X) \cong \mathsf{D}(Y)$*, then*

$$\underline{J}_a^3(X_{\mathbb{C}}) \sim \underline{J}_a^3(Y_{\mathbb{C}}).$$

Variant

If K an arbitrary perfect field, replace $J_a^3(X)$ with a distinguished model of $\mathrm{Ab}^2(X_{\overline{K}})$, Murre's algebraic representative for $\mathrm{A}^2(X_{\overline{K}})$.

Thanks!